VI.—*On the Analysis of Discontinuous Functions.* *By* George Boole, Esq.

Read 20th July, 1846.

THE design of this paper is to illustrate the mathematical doctrine of discontinuity, and to present a few remarks on some connected subjects. We shall first deduce in succession three theorems, by which the function $f(x)$ may be expressed in subjection to any proposed form of discontinuity. The last of these theorems will be identical with Fourier's, and the second, and probably the first, are known; but the connexion in which they will be presented is, perhaps, new, and may not be devoid of interest. Subsequently, a theorem will be deduced for the discontinuous expression of $\frac{f(x)}{t^n}$, which admits of very interesting applications to the theory of definite multiple integrals. One such application will form the subject of another paper. As respects the theorem of Fourier, usually expressed in the form

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} da\,dv \cos(av - xv) f(a), \tag{1}$$

we shall endeavour to show that the right hand member is to be considered as nothing else than an abbreviation, sanctioned by custom, of the expression

$$\text{limit of } \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} da\,dv\, \epsilon^{-kv} \cos(av - xv) f(a), \tag{2}$$

k approaching through positive values to 0; and that if this meaning is neglected,—if the sign of integration relative to v is taken in its ordinary import, we have no assurance of the truth of the theorem. It might appear to be superfluous to insist on a position like this, on which the testimony of some writers is

so clear and explicit, but that, in point of fact, the views which are very commonly held on this subject are vague and unsatisfactory, as recent discussions have tended to show. A hope is entertained that the considerations by which the theorem is here deduced will serve to render its real nature less doubtful. In a supplementary chapter we shall, by way of further illustration, adduce some examples of integrals, which are, in like manner, to be considered as the limits to which more general integrals approach, as two quantities, k and k', approach to 0. This will throw some light on the cause of the difference which is sometimes observed between series and their envelopments, when submitted to a process of definite integration, and will also explain the appearance of the imaginary factor in integrals, the subjects of which become infinite within the limits of integration. We shall conclude this part of the subject by an application, which, though it has no special bearing on the principles above stated, may be thought to possess an independent interest. By the aid of Fourier's theorem, from the equation $f(u) = x$ will be deduced a symbolical expression for the value of $\mathrm{F}(u)$, involving an arbitrary function of x, independent of the form of either $f(u)$ or $\mathrm{F}(u)$. Particular determinations of this function will lead us to the known theorems of Lagrange and Laplace, while its arbitrary character will show that these belong to a class of theorems of which the number is infinite.

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1. Those who, commencing the study of the differential calculus at a mature age, bring to it a mind disciplined by the pursuit of other sciences, feel an insuperable difficulty in admitting, what some writers assert as an axiom or first principle, that what is true up to the limit is true at the limit. And doubtless this is an unsound induction. The only condition under which we are permitted to pass from a series of particular truths to some general proposition is, that every truth included in the general may be proved in the particular; unless, indeed, we assert an *a priori* principle of continuity, antecedent to all experience, and independent of all proof, which were to abandon the dominion of reason for that of intuition. But not only is the axiom alluded to unfounded,—it is unnecessary to the purpose which it is thought to subserve. The differential calculus is the calculus of limits, not of values actually realized. What is, quantitatively, the *value* of a

vanishing fraction under limiting circumstances, we cannot determine from its value under other circumstances. Probably it ceases to be a subject of value. But the limit to which the series of its known values approximates, is something which we may determine, and to which we are able to attach a definite meaning: and for all applications this is sufficient. In one form or other the conception of a limit is indispensable; but, justly considered, this is a new subject of thought, not the basis of a new condition of reasoning.

In the integral calculus such general considerations may be applied to the case of those formulæ of definite integration, the evaluation of which is made to depend on the assumption that some quantity, k, which they involve, is positive. We may conceive of such formulæ as tending to limits, and of theorems in which the consideration of those limits is involved. Fourier's theorem, as has been already remarked, is of this class. Our purpose will be accomplished if we shall establish this position, and prove its sufficiency by further illustrations.

2. If in the function $\tan^{-1}\dfrac{a-x}{k}$, we suppose $x < a$, and k a positive quantity, then, as k is diminished, the limit of the values of the function will be $\dfrac{\pi}{2}$. This is evident.

If $x = a$, the limit is 0, the entire series of values being 0.

If $x > a$, the limit is $-\dfrac{\pi}{2}$.

Let

$$\Delta f(a) = f(a+\Delta a) - f(a), \tag{3}$$

then

$$\Delta \tan^{-1}\frac{a-x}{k} = \tan^{-1}\frac{a+\Delta a-x}{k} - \tan^{-1}\frac{a-x}{k}; \tag{4}$$

and applying what precedes to each term of the second member, we find that the limit of their sum is π, or $\dfrac{\pi}{2}$, or 0, according as x lies between a and $a+\Delta a$, or is equal to a or $a+\Delta a$, or lies entirely without those limits.

In what follows we shall suppose that k is thus diminished, so that by any expression involving k we shall understand the limit to which it approaches as k approaches to 0. Then

$$\frac{1}{\pi}\left(\tan^{-1}\frac{a+\Delta a-x}{k} - \tan^{-1}\frac{a-x}{k}\right)f(x) = f(x) \text{ or } \frac{f(x)}{2} \text{ or } 0, \tag{5}$$

according as x lies between, upon, or without the limits a and $a+\Delta a$. If we wish that $f(x)$ should vanish for values of x external to p and q, we shall have

$$\frac{1}{\pi}\left\{\tan^{-1}\left(\frac{p-x}{k}\right)-\tan^{-1}\frac{q-x}{k}\right\}f(x)=f(x) \text{ or } \frac{f(x)}{2} \text{ or } 0, \tag{6}$$

according as x lies within, upon, or without the limits p and q.

As regards the mere expression of discontinuity, this formula is on a par with those which will follow, but it is wanting in certain other advantages which they possess.

3. When Δa becomes infinitesimal it may be replaced by da, and the symbol Δ by d, whence, by (4) and its consequences,

$$d\tan^{-1}\frac{a-x}{k}=\pi \text{ or } \frac{\pi}{2} \text{ or } 0, \tag{7}$$

according as x lies between, upon, or without the limits a and $a+da$. Effecting the operation in the first member, we have, under the same conditions,

$$\frac{kda}{k^2+(a-x)^2}=\pi \text{ or } \frac{\pi}{2} \text{ or } 0;$$

and since, under the two first conditions, the values of a and x are indefinitely near to each other,

$$\frac{1}{\pi}\frac{kdaf(a)}{k^2+(a-x)^2}=f(x) \text{ or } \frac{f(x)}{2} \text{ or } 0. \tag{8}$$

Extend this by integration from p to q, then observing that each half value, $\frac{f(x)}{2}$, occurs in two contiguous elements, except the first of them and the last, we have

$$\frac{1}{\pi}\int_p^q\frac{kdaf(a)}{k^2+(a-x)^2}=f(x) \text{ or } \frac{f(x)}{2} \text{ or } 0, \tag{9}$$

according as x lies within, upon, or without the limits p and q.

If p and q are respectively $-\infty$ and ∞, then, for all real and finite values of x,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{kdaf(a)}{k^2+(a-x)^2}=f(x). \tag{10}$$

As regards the expression of discontinuity, this formula is equivalent to (6); but we have gained this important point, that x only enters the first member in a rational form.

The general formula (9) may be verified thus, for the particular case of $f(a) = a^n$,

$$\frac{ka^n da}{k^2+(a-x)^2} = \left\{ka^{n-2} + \frac{2kxa^{n-1} - kx^2a^{n-2} - k^3a^{n-2}}{k^2+(a-x)^2}\right\} da.$$

Let $\int_p^q \frac{ka^n da}{k^2+(a-x)^2} = u_n$, then the above equation gives, on integration,

$$u_n = \frac{1}{\pi}\int_p^q ka^{n-2}\,da + 2xu_{n-1} - x^2u_{n-2} - k^2u_{n-2},$$

which, when k becomes infinitesimal, gives

$$u_n - 2xu_{n-1} + x^2u_{n-2} = 0,$$

an equation of differences, of which the solution is

$$u_n = (c + c'n)x^n.$$

To determine the constants, we have

$$u_0 = c,$$
$$u_1 = (c+c')x.$$

Hence

$$c = u_0, \quad c' = \frac{u_1}{x} - u_0,$$

therefore

$$u_n = \left(u_0 + \frac{u_1 n}{x} - u_0 n\right)x^n. \tag{11}$$

Now

$$u_0 = 1, \text{ or } \tfrac{1}{2}, \text{ or } 0,$$
$$u_1 = x, \text{ or } \frac{x}{2}, \text{ or } 0,$$

according as x lies within, upon, or without the limits p and q. Substituting in (11) we have in the first case $u_n = x^n$, in the second $u_n = \frac{x^n}{2}$, in the third $u_n = 0$ as we should expect.

4. Resuming the formula

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{kdaf(a)}{k^2+(a-x)^2},$$

and observing that

$$\frac{k}{k^2+(a-x)^2}=\tfrac{1}{2}\left\{\frac{1}{k+(a-x)\sqrt{-1}}+\frac{1}{k-(a-x)\sqrt{-1}}\right\}$$
$$=\tfrac{1}{2}\left\{\int_0^{\infty}\epsilon^{-(k+(a-x)\sqrt{-1})v}dv+\int_0^{\infty}\epsilon^{-(k-(a-x)\sqrt{-1})v}dv\right\}$$
$$=\int_0^{\infty}\epsilon^{-kv}\cos(av-xv)\,dv,$$

we have, on substitution,

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}dadv\,\epsilon^{-kv}\cos(av-xv)f(a), \tag{12}$$

which may be written in the form

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}dadv\cos(av-xv)f(a), \tag{13}$$

provided that the second member be regarded as an abbreviated form of the second member of the equation preceding; according to which assumption $\int_0^{\infty}dv\phi(v)$ is to be understood to mean the limit of $\int_0^{\infty}dv\,\epsilon^{-kv}\phi(v)$ for positive diminishing values of k. We may call this the extraordinary, or limiting meaning of the symbol $\int$.

But is the formula (13) true, when to the second sign of integration we attach its ordinary meaning? If it is true, it must be so either in virtue of the principle of continuity assumed *à priori*, or because the proposition which is affirmed of the integrals is true universally of their elements. The former assumption we dismiss; the latter requires us to consider whether the limit of $\epsilon^{-kv}\cos(av-xv)$ is $\cos(av-xv)$, for all values of v from 0 to ∞ included. It is so for all finite values of v; but when v is infinite, the limit of the first expression is 0, since the value of $\epsilon^{-kv}\cos(av-kv)$, for every positive value of k, is nothing. But the second expression becomes $\cos\infty$, respecting which, if we were entitled to make any assertion, it would be that it is indefinite. When a definite value is assigned to $\cos\infty$, it is through that assumption of continuity of which we have already spoken, as involving an unauthorized induction. We

conclude, therefore, that the second symbol of integration in Fourier's theorem is not to be taken in its ordinary acceptation.

The peculiar advantage of Fourier's theorem over (5) and (9) is, that x enters into the second member only exponentially, through the cosine.

5. Resuming the equation

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}da\,dv\cos(av-xv)f(a),$$

and writing the cosine in its exponential form, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}da\,dv\{\epsilon^{(av-xv)\sqrt{-1}}+\epsilon^{-(av-xv)\sqrt{-1}}\}f(a). \qquad (14)$$

Now

$$\frac{1}{t^n}=\frac{\epsilon^{n\frac{\pi}{2}\sqrt{-1}}}{\Gamma(n)}\int_{0}^{\infty}dw\,w^{n-1}\epsilon^{-tw\sqrt{-1}}=\frac{\epsilon^{-n\frac{\pi}{2}\sqrt{-1}}}{\Gamma(n)}\int_{0}^{\infty}dw\,\epsilon^{tw\sqrt{-1}}w^{n-1}, \qquad (15)$$

the symbol $\int_0^\infty$ being used in its extraordinary or limiting sense. Multiply the successive terms of (14) by those of (15) respectively, and we have

$$\frac{f(x)}{t^n}=\frac{1}{2\pi\Gamma(n)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}da\,dv\,dw\,\{\epsilon^{(av-xv-tw+\frac{n\pi}{2})\sqrt{-1}}+\epsilon^{-(av-xv-tw+\frac{n\pi}{2})\sqrt{-1}}\}\,w^{n-1}f(a),$$

and converting the exponentials into cosines and sines,

$$\frac{f(x)}{t^n}=\frac{1}{\pi\Gamma(n)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}da\,dv\,dw\cos\left(av-xv-tw+\frac{n\pi}{2}\right)w^{n-1}f(a). \qquad (16)$$

Of course this theorem only enables us to represent discontinuity relative to x. Its advantage, with reference to t, is, that it introduces that variable under the symbol cosine. The application is reserved for a separate paper.

SUPPLEMENTARY INVESTIGATIONS.

1. We have designated as extraordinary or limiting integrals those which are to be considered as the limits to which others approximate, as some quantity approaches to 0. Of this kind is the integral $\int_0^\infty dx\cos(qx)x^{n-1}$, n being positive,

which is to be regarded as the limit of the more general integral $\int_0^\infty dx\,\epsilon^{-kx}\cos(qx)x^{n-1}$, k being positive, the value of the former being, on this assumption,

$$\int_0^\infty dx\cos(qx)x^{n-1}=\frac{\Gamma(n)}{q^n}\cos\left(n\frac{\pi}{2}\right). \qquad (1)$$

We are hence naturally led to the consideration of such integrals as constitute the limiting cases of more general forms, involving two vanishing quantities, k and k'. Of this kind would be the integral above mentioned, when n should be negative, as may be shewn by the following analysis. We have

$$\int_0^\infty dv\,\epsilon^{-kv}\cos(xv)v^{n-1}=\frac{\Gamma(n)}{(k^2+x^2)^{\frac{n}{2}}}\cos\left(n\tan^{-1}\frac{x}{k}\right) \qquad (2)$$

$$\therefore\ \frac{\cos\left(n\tan^{-1}\frac{x}{k}\right)}{(k^2+x^2)^{\frac{n}{2}}}=\frac{1}{\Gamma(n)}\int_0^\infty dv\,\epsilon^{-kv}\cos(xv)v^{n-1},$$

and

$$\int_0^\infty\frac{dx\,\epsilon^{-k'x}\cos qx\cos\left(n\tan^{-1}\frac{x}{k}\right)}{(k^2+x^2)^{\frac{n}{2}}}=\frac{1}{\Gamma(n)}\int_0^\infty\int_0^\infty dx\,dv\,\epsilon^{-kv-k'x}\cos(qx)\cos(vx)v^{n-1}$$

$$=\frac{\pi}{2\Gamma(n)}(\mathrm{u}+\mathrm{v}); \qquad (3)$$

provided that

$$\mathrm{u}=\frac{1}{\pi}\int_0^\infty\int_0^\infty dv\,dx\,\epsilon^{-k'x}\cos(vx-qx)\,\epsilon^{-kv}v^{n-1},$$

$$\mathrm{v}=\frac{1}{\pi}\int_0^\infty\int_0^\infty dv\,dx\,\epsilon^{-k'x}\cos(vx+qx)\,\epsilon^{-kv}v^{n-1},$$

expressions of which the limiting values, with reference to k', are, by Fourier's theorem,

$$\mathrm{u}=\epsilon^{-kq}q^{n-1}\text{ or }0,\text{ as }q>0,\text{ or }q<0,$$

$$\mathrm{v}=\epsilon^{kq}(-q)^{n-1}\text{ or }0,\text{ as }-q>0,\text{ or }-q<0,$$

$$\text{or as }q<0,\text{ or }q>0,$$

$$\therefore\ \mathrm{u}+\mathrm{v}=\epsilon^{-kq}q^{n-1},\text{ or }\epsilon^{kq}(-q)^{n-1}\text{ as }q>0,\text{ or }q<0.$$

Hence, taking the limit with reference to k, and substituting in (3). Limit of

$$\int_0^\infty\frac{dx\,\epsilon^{-k'x}\cos(qx)\cos\left(n\tan^{-1}\frac{x}{k}\right)}{(k^2+x^2)^{\frac{n}{2}}}=\frac{\pi}{2\Gamma(n)}q^{n-1},\text{ or }\frac{\pi}{2\Gamma(n)}(-q)^{n-1}, \qquad (4)$$

as q is positive or negative. Now if, under the integral sign in the first member, we make k' and k to vanish, we have

$$\int_0^\infty \frac{dx \cos(qx)\cos\frac{n\pi}{2}}{x^n} = \frac{\pi}{2\Gamma(n)}(\pm q)^{n-1}$$

$$\therefore \int_0^\infty \frac{dx \cos qx}{x^n} = \frac{\pi(\pm q)^{n-1}}{2\Gamma(n)\cos\frac{n\pi}{2}}, \tag{5}$$

the upper or lower sign being taken, according as q is positive or negative. The first member of this equation is a limiting integral of the second class, and is to be regarded as merely an abbreviation of the first member of (4). Taken in any other sense, the equation is certainly not true.

If in (1) we make $q = 1$, we have

$$\int_0^\infty dx \cos(x) x^{n-1} = \Gamma(n)\cos\left(n\frac{\pi}{2}\right),$$

$$\therefore \Gamma(n) = \frac{1}{\cos\left(n\frac{\pi}{2}\right)}\int_0^\infty dx\cos(x)x^{n-1};$$

and if we assume this as the universal definition of $\Gamma(n)$, and regard the integral in the second member as a limiting one of the first or second class, according as n is positive or negative, we may deduce the law of continuity of $\Gamma(n)$. For compare the first members of (1) and (5) with the general form $\int_0^\infty dx\cos(x)x^{l-1}$, in which l may be positive or negative; we have, in the former case, $q = 1$, $n = l$, and the equation (1) becomes

$$\int_0^\infty dx \cos(x)\, x^{l-1} = \Gamma(l)\cos\left(l\frac{\pi}{2}\right);$$

in the latter case, $q = 1$, $n = 1 - l$, and the equation (5) becomes

$$\int_0^\infty dx\cos(x)\,x^{l-1} = \frac{\pi}{2\Gamma(1-l)\cos\left((1-l)\frac{\pi}{2}\right)} = \frac{\pi}{2\Gamma(1-l)\sin\left(l\frac{\pi}{2}\right)}.$$

Whence

$$\Gamma(l)\cos\left(l\frac{\pi}{2}\right) = \frac{\pi}{2\Gamma(1-l)\sin\left(l\frac{\pi}{2}\right)}$$

$$\therefore \Gamma(l)\Gamma(1-l)=\frac{\pi}{2\cos\left(l\frac{\pi}{2}\right)\sin\left(l\frac{\pi}{2}\right)}=\frac{\pi}{\sin(l\pi)}, \tag{6}$$

a theorem which is known to be true of Γ in its ordinary definition, when l lies between 0 and 1, but which, according to the definition of Γ adopted in this investigation, is thus seen to be true universally.

2. The circumstance of an integral becoming infinite between the limits of integration, gives rise, as is well known, to an imaginary term in the function expressing its value. When this term is rejected according to Cauchy's rule, a discontinuity of form is observed between the cases of the integral here considered, and those in which the integral does not become infinite between the limits. Under these circumstances the former of these two cases may be regarded as a limiting one, and of different origin from the other, and this is the true explanation of the discontinuity. Thus (l positive),

$$\int_0^\infty \frac{dx\cos(lx)}{(x^2+h^2)}=\frac{\pi}{2h}\epsilon^{-lh},\quad \int_0^\infty \frac{dx\cos(lx)}{x^2-h^2}=-\frac{\pi}{2h}\sin(lh). \tag{7}$$

But if, in the first theorem, we change h into $h\sqrt{-1}$, we have

$$\int_0^\infty \frac{dx\cos(lx)}{x^2-h^2}=-\frac{\pi}{2h}\sin(lh)-\sqrt{-1}.\frac{\pi}{2h}\cos(lh),$$

the second member of which includes the rejected term. By an analysis founded on Fourier's theorem we find, however, that the two integrals in (7) are respectively limits of the following,

$$\int_0^\infty \frac{dx\,\epsilon^{-kx}\cos(lx)}{(x^2+h^2)}\text{ and }\int_0^\infty dx\,\epsilon^{-kx}\cos lx\,\frac{x^2-k^2-k'^2}{\{(x+h)^2+k'^2\}\{(x-h)^2+k'^2\}},$$

k and k' vanishing.

All these circumstances require to be noticed when we compare a definite integral with its development. Thus the second theorem of (7) may be obtained by developing in descending powers of x, and integrating each term separately by (5); but it cannot be obtained from the ascending development. Although the reason, in this case, is obvious, we are not prepared to lay down any general rule, but we conceive that such a rule must recognise the distinctions which we have endeavoured to elucidate.

CONNEXION OF FOURIER'S THEOREM WITH THE THEORY OF EQUATIONS.

1. Theorem.—The value, v, of the definite double integral,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da\,dv\,\epsilon^{(a-x)v\sqrt{-1}} f(a, v\sqrt{-1}), \tag{1}$$

is symbolically expressed by the equation

$$\mathrm{v} = \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} f(x, \epsilon^{\theta}), \tag{2}$$

provided that, after expansion and differentiation, we assume $\theta = 0$.

For if $\theta = 0$,

$$\begin{aligned} f(a, v\sqrt{-1}) &= f(a, v\sqrt{-1}\,\epsilon^{\theta}) \\ &= f(a, \epsilon^{\theta + \log v\sqrt{-1}}) \\ &= \epsilon^{\log v\sqrt{-1}\frac{d}{d\theta}} f(a, \epsilon^{\theta}), \text{ by Taylor's theorem,} \\ &= (v\sqrt{-1})^{\frac{d}{d\theta}} f(a, \epsilon^{\theta}). \end{aligned} \tag{3}$$

Hence

$$\mathrm{v} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da\,dv\,\epsilon^{(a-x)v\sqrt{-1}}(v\sqrt{-1})^{\frac{d}{d\theta}} f(a, \epsilon^{\theta}). \tag{4}$$

Now

$$\epsilon^{(a-x)v\sqrt{-1}}(v\sqrt{-1})^{\lambda} = \left(-\frac{d}{dx}\right)^{\lambda} \epsilon^{(a-x)v\sqrt{-1}}$$

when λ is a constant quantity; and if, in the right hand member of (4), we suppose $f(a, \epsilon^{\theta})$ to be developed in descending powers of ϵ^{θ}, then in each term of the expansion, $\frac{d}{d\theta}$ will be represented by a constant quantity, by virtue of the known theorem,

$$\phi\left(\frac{d}{d\theta}\right)\epsilon^{n\theta} = \phi(n)\epsilon^{n\theta}.$$

Hence

$$\epsilon^{(a-x)v\sqrt{-1}}(v\sqrt{-1})^{\frac{d}{d\theta}} f(a, \epsilon^{\theta}) = \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} \epsilon^{(a-x)v\sqrt{-1}} f(a, \epsilon^{\theta}),$$

and substituting in (4),

$$v = \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dadv \, \epsilon^{(a-x)v\sqrt{-1}} f(a, \epsilon^{\theta})$$

$$= \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} f(x, \epsilon^{\theta}), \qquad (5)$$

by a known form of Fourier's theorem.

Since, in this investigation, the function $f(a, \epsilon^{\theta})$ may be discontinuous, and may be supposed to vanish for all values of a which are not included between given limits, it is obvious that the value of the more general integral,

$$v = \frac{1}{2\pi} \iint_{-\infty}^{\infty} dadv \, \epsilon^{(a-x)v\sqrt{-1}} f(a, \epsilon^{\theta}), \qquad (6)$$

will be expressed by the formula

$$v = \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} f(x, \epsilon^{\theta}),$$

when x lies within those limits, supposed to be real, to which the integration relative to a extends, and that it will be 0 when x lies entirely without those limits.

2. The above premised, let us consider the expression

$$v = \frac{1}{2\pi} \iint_{-\infty}^{\infty} dadv \, \epsilon^{(f(a)-x)v\sqrt{-1}} \, F(a), \qquad (7)$$

which may be written in the form

$$v = \frac{1}{2\pi} \iint_{-\infty}^{\infty} dadv \, \epsilon^{(\phi^{-1}(a)-x)v\sqrt{-1}} \, \epsilon^{(f(a)-\phi^{-1}(a))v\sqrt{-1}} F(a),$$

$\phi^{-1}(a)$ denoting an arbitrary function of a, to which the inverse form is here given for the convenience of a subsequent transformation.

Let $\phi^{-1}(a) = a'$, then $a = \phi(a')$, $da = \phi'(a')da'$, therefore

$$v = \frac{1}{2\pi} \iint_{-\infty} da'dv \, \epsilon^{(a'-x)v\sqrt{-1})} \, \epsilon^{(f[\phi(a')]-a')v\sqrt{-1}} \, F[\phi(a')] \phi'(a');$$

the value of which, supposing the limits of a' to be real, and x to lie within those limits, is

$$v = \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} \epsilon^{(f[\phi(x)]-x)\epsilon^{\theta}} \, F[\phi(x)]\phi'(x), \qquad (8)$$

the function $\phi(x)$ being arbitrary.

Now let $\phi(x) = f^{-1}(x)$, then $f[\phi(x)] - x = 0$, and $\epsilon^{(f[\phi(x)]-x)\epsilon^{\theta}} = 1$, therefore

$$\mathrm{V} = \mathrm{F}[f^{-1}(x)] \frac{d}{dx} f^{-1}(x).$$

Suppose that $f(u) = x$, then

$$\mathrm{V} = \mathrm{F}(u) \frac{du}{dx};$$

wherefore

$$\mathrm{F}(u) \frac{du}{dx} = \left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}} \epsilon^{(f[\phi(x)]-x)\epsilon^{\theta}} \mathrm{F}[\phi(x)]\phi'(x). \qquad (9)$$

The transformation of the original integral V, by which the first member of this equation was obtained, was virtually $a = f^{-1}(a')$, and that by which the second member was obtained was $a = \phi(a')$, and in both cases a' must be real. This condition requires that a' and $f[\phi(a')]$ should be both real; and as the limits of a' determine those of x, it is necessary that x and $f[\phi(x)]$ should be real. Apparently this is the only condition to which $\phi(x)$ is subject.

From (9) we have

$$\left(\frac{d}{dx}\right)^{-1} \mathrm{F}(u) \frac{du}{dx} = -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{(f\phi[(x)]-x)\epsilon^{\theta}} \mathrm{F}[\phi(x)]\phi'(x).$$

Let

$$\left(\frac{d}{dx}\right)^{-1} \mathrm{F}(u) \frac{du}{dx} = \mathrm{F}_{,}(u),$$

then

$$\mathrm{F}[\phi(x)]\phi'(x) = \mathrm{F}'_{,}[\phi(x)]\phi'(x).$$

Substituting, and dropping the suffix,

$$\mathrm{F}(u) = -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{(f[\phi(x)]-x)\epsilon^{\theta}} \mathrm{F}'\phi(x)\phi'(x), \qquad (10)$$

u being a root of the equation

$$f(u) = x,$$

and $\phi(x)$ an arbitrary function of x, rendering $f[\phi(x)]$ real.

3. In order to apply the above theorem, let us write for simplicity,

$$\mathrm{X} = x - f[\phi(x)],$$

then

$$\mathrm{F}(u) = -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{-\mathrm{X}\epsilon^{\theta}} \mathrm{F}'[\phi(x)]\phi'(x)$$

$$= -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \left(1 - \mathrm{X}\epsilon^{\theta} + \frac{\mathrm{X}^2}{1.2}\epsilon^{2\theta} - \frac{\mathrm{X}^3}{1.2.3}\epsilon^{3\theta} . .\right) \mathrm{F}'[\phi(x)]\phi'(x)$$

$$= \mathrm{F}[\phi(x)] + \mathrm{X}\mathrm{F}'[\phi(x)]\phi'(x) + \frac{1}{1.2}\frac{d}{dx}\mathrm{X}^2\mathrm{F}'[\phi(x)]\phi'(x) + \frac{1}{1.2.3}\frac{d^2}{dx^2}\mathrm{X}^3\mathrm{F}'[\phi(x)]\phi'(x), \quad (11)$$

by virtue of the relation $\phi\left(\frac{d}{d\theta}\right)\epsilon^{n\theta} = \phi(n)\epsilon^{n\theta}$.

The theorems of Lagrange and Laplace for the expansion of functions are particular cases of the above, which, it is seen, includes an infinite diversity of particular theorems, corresponding to the different forms which may be attributed to $\phi(x)$. To deduce Lagrange's theorem, we have

$$u = x + hf(u).$$
$$\therefore u - hf(u) = x.$$

Hence in (11) writing $u - hf(u)$ in the place of $f(u)$, or $x - hf(x)$ in that of $f(x)$, we have

$$\mathrm{X} = x - \phi(x) + hf[\phi(x)].$$

Let $\phi(x) = x$, then $\mathrm{X} = hf(x)$, and therefore

$$\mathrm{F}(u) = \mathrm{F}(x) + hf(x)\mathrm{F}'(x) + \frac{h^2}{1.2}\frac{d}{dx}f(x)^2\mathrm{F}'(x) \ldots \quad (12)$$

which is Lagrange's theorem; but every other form of $\phi(x)$, which makes $\phi(x) - hf[\phi(x)]$ real, and therefore, at least, every real form of $\phi(x)$, will give a true expansion.

To deduce Laplace's theorem, we have

$$u = \psi(x + hf(u)), \quad \therefore \psi^{-1}(u) - hf(u) = x.$$

Here, therefore, $\psi^{-1}(x) - hf(x)$ must be written for $f(x)$, whence

$$\mathrm{X} = x - \psi^{-1}[\phi(x)] + hf[\phi(x)].$$

Assume $\phi(x) = \psi(x)$, then $\mathrm{X} = hf[\psi(x)]$, and substituting in (11)

$$\mathrm{F}(u) = \mathrm{F}[\psi(x)] + hf[\psi(x)]\mathrm{F}'[\psi(x)]\psi'(x) + \frac{h^2}{1.2}\frac{d}{dx}f[\psi(x)]^2\mathrm{F}'\psi(x)\psi'(x), \quad (13)$$

which is but one of an infinite number of possible developments.

If, in (10), we make $\phi(x) = x$, we have

$$\mathrm{F}(u) = -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{(f(x)-x)\epsilon^{\theta}}\,\mathrm{F}'x\,; \tag{14}$$

and it may be interesting to observe, that the general theorem may be proved from the particular one in the following manner:

Let $u = \phi(v)$, then, since $f(u) = x$, we have $f[\phi(v)] = x$. Hence, in (14), changing u into v, and writing $f\phi$ for f, and $\mathrm{F}\phi$ for F, we have

$$\mathrm{F}[\phi(v)] = -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{(f[\phi(x)]-x)\epsilon^{\theta}}\,\mathrm{F}'[\phi(x)]\phi'(x)\,;$$

or,

$$\mathrm{F}u = -\left(-\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{(f[\phi(x)]-x)\epsilon^{\theta}}\,\mathrm{F}'\phi(x)\phi'(x).$$

In this way the general theorems of this paper might have been deduced from the particular theorem of Lagrange, but it would have been difficult to foresee such a generalization, until, by other methods, it had been shewn to be possible: while the connexion of these results with Fourier's theorem is in itself a very interesting fact.

4. As a final illustration of the general theorem (11), let it be supposed that we are in possession of the solution of the equation $f(u) = x$, which we shall represent by $u = v$, and that we desire to express the solution of the equation

$$f(u) + f_{\prime}(u) = x$$

in a series which shall converge rapidly when $f_{\prime}(v)$ is small.

Writing in (11) u for $\mathrm{F}(u)$, and $f(u) + f_{\prime}(u)$ for $f(u)$, we have

$$\mathrm{X} = x - f(\phi(x)) - f_{\prime}(\phi(x)).$$

Let $x - f(\phi(x)) = 0$, then $\phi(x) = v$, and $\mathrm{X} = -f_{\prime}(v)$; therefore,

$$u = v - f_{\prime}(v)\frac{dv}{dx} + \frac{1}{1.2}\frac{d}{dx}f_{\prime}(v)^2\frac{dv}{dx} - \frac{1}{1.2.3}\frac{d^2}{dx^2}f_{\prime}(v)^3\frac{dv}{dx}, \text{ \&c.} \tag{15}$$

It may be worth while to observe, in conclusion, that the general theorem (10) may be expressed in the somewhat more convenient form

$$\mathrm{F}(u) = \left(\frac{d}{dx}\right)^{\frac{d}{d\theta}-1} \epsilon^{(x-f[\phi(x)])\epsilon^{\theta}}\,\mathrm{F}'[\phi(x)]\phi'(x), \tag{16}$$

as is evident on expansion.

If $F(u) = u$, we obtain from the general theorem that root of the equation $f(u) = x$, which is represented by $\phi(v)$, where v is the least root of the equation

$$f[\phi(v)] = x.$$

Methods of determining the form of ϕ, so as to accomplish particular objects, will suggest themselves to the reader.